Quantized Vortices in the Ideal Bose Gas: A Physical Realization of Random Polynomials

Yvan Castin,[1] Zoran Hadzibabic,[1] Sabine Stock,[1] Jean Dalibard,[1] and Sandro Stringari[1,2,3]
[1]*Laboratoire Kastler Brossel, Ecole normale supérieure, 24 rue Lhomond, 75005 Paris, France*
[2]*Collège de France, 11 place Marcelin Berthelot, 75005 Paris, France*
[3]*Dipartimento di Fisica, Università di Trento and CNR-INFM BEC Center, I-38050 Povo, Trento, Italy*
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We propose a physical system allowing one to experimentally observe the distribution of the complex zeros of a random polynomial. We consider a degenerate, rotating, quasi-ideal atomic Bose gas prepared in the lowest Landau level. Thermal fluctuations provide the randomness of the bosonic field and of the locations of the vortex cores. These vortices can be mapped to zeros of random polynomials, and observed in the density profile of the gas.

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An important field of study in theoretical statistical physics concerns the properties of the roots of random polynomials [1,2]. Of particular interest is the so-called Weyl polynomial for the complex variable ζ:

$$P(\zeta) = \sum_{m=0}^{m_{\max}} a_m \frac{\zeta^m}{\sqrt{m!}}, \qquad (1)$$

where the a_m's are independent random complex numbers with the same Gaussian probability distribution. The roots of P in the complex plane can be mapped to a two-dimensional (2D) gas of particles with repulsive interactions. They are spatially antibunched, and have a uniform mean density in the large $m_{\max}$ limit [3].

Although the statistical properties of the roots of the Weyl polynomial have been well studied theoretically, no physical system has allowed yet to observe them directly. In this Letter we show that a 2D rotating ideal Bose gas is a well-suited system for this observation. The positions of the vortices appearing in the gas can be mapped to the zeros of the random polynomial describing the atomic state. More precisely, the gas is harmonically trapped and it rotates at a frequency close to the trapping frequency so that it is "frozen" in the lowest Landau level (LLL) [4–6]. Its finite temperature T ensures that several vortices are present in the region where atomic density is significant, and thermal fluctuations provide the randomness of the vortex locations in different realizations of the experiment. Such an experiment is not unrealistic: a 2D atomic Bose gas in the LLL has recently been produced [7], and the use of a Fano-Feshbach scattering resonance allows the near cancellation of the interactions between ultracold atoms [8]. This regime of an ideal gas with large thermal fluctuations dramatically differs from the well-studied case of a rotating and interacting Bose-Einstein condensate in the LLL at $T = 0$, where the vortices are known to form an Abrikosov lattice [4,5,7].

The ideal gas in our model is confined in a harmonic trap, with oscillation frequency ω in the xy plane and ω_z along z. The confinement along z is assumed to be strong, $k_B T \ll \hbar\omega_z$, so that the z degree of freedom is frozen and the gas is kinematically two dimensional. We also assume that some angular momentum has been transferred to the gas by a stirring procedure [9], so that thermal equilibrium is reached in a frame rotating at frequency Ω around z. Ω is chosen close to the trapping frequency ω:

$$\omega - \Omega \ll \omega, \qquad (2)$$

which is experimentally realistic since the value $(\omega - \Omega)/\omega = 0.01$ has already been achieved [7]. We also assume that the gas is cooled to a low enough temperature, $k_B T \ll 2\hbar\omega$, so that the relevant single particle states are linear combinations of the LLL eigenmodes [4,5]:

$$\phi_m(x, y) = \frac{\zeta^m}{\sqrt{\pi m!}} e^{-\zeta\zeta^*/2}, \qquad \zeta = x + iy. \qquad (3)$$

Here $a_{\text{ho}} = \sqrt{\hbar/M\omega}$ (where M is the atomic mass) is taken as the unit of length. The mode energy $\epsilon_m = m\hbar(\omega - \Omega)$ depends on a single quantum number $m \geq 0$ so that thermodynamically the gas is effectively one dimensional.

The relevant quantity in our study is the density of vortices, which we will define in relation with the complex classical field $\psi(\mathbf{r})$ describing the state of the gas. This classical field represents not only the state of the ground mode ϕ_0 (where a vortex-free condensate forms at low enough temperature), but also the states of all other modes. We obtain ψ using the expression for the many-body density operator $\hat{\sigma}$ of the ideal gas in thermal equilibrium in the grand canonical ensemble [10]:

$$\hat{\sigma} = \int \mathcal{D}\psi P(\{\psi\}) |\text{coh: } \psi\rangle\langle\text{coh: } \psi|. \qquad (4)$$

In this expression $\hat{\sigma}$ is a statistical mixture of Glauber coherent states $|\text{coh:}\psi\rangle$, with positive weights $P(\{\psi\})$ (the so-called Glauber P distribution) given by the Gaussian functional specified below. A given realization of the experiment can then be viewed as a random draw of the atomic field state ψ, with the probability law $P(\{\psi\})$.

The stochastic nature of ψ is simple to characterize by expanding it on the eigenmodes ϕ_m:

$$\psi(\mathbf{r}) = \sum_{m\geq 0} a_m \phi_m(\mathbf{r}). \quad (5)$$

The a_m's are complex, statistically independent random numbers, with a Gaussian law:

$$P(\{\psi\}) \propto \prod_{m\geq 0} e^{-|a_m|^2/n_m}, \quad (6)$$

where $n_m = [\exp(\beta(\epsilon_m - \mu)) - 1]^{-1}$ is the mean occupation number of mode m. Here $\beta = 1/k_B T$ and μ is the chemical potential. This provides a clear link with the random polynomial of Eq. (1) when several n_m have similar values:

$$\psi(\mathbf{r}) = f(\zeta)\frac{e^{-\zeta\zeta^*/2}}{\sqrt{\pi}}, \qquad f(\zeta) = \sum_m a_m \frac{\zeta^m}{\sqrt{m!}}. \quad (7)$$

When $f(\zeta)$ is factorized as $f(\zeta) \propto \prod_i(\zeta - \zeta_i)$, each root ζ_i corresponds to the location of a positively charged vortex in the field ψ. Since having a multiple root is a zero measure event, these vortices are of charge unity. Note that ψ results from the interference of a large number of macroscopically populated field modes, reminiscent of the interference of independent condensates [11,12].

The standard case where all the a_m's have the same variance corresponds in our model to all ϵ_m's being equal, i.e., $\Omega = \omega$. The average vortex density is then uniform, $\bar{\rho}_v = 1/\pi$ and the average pair distribution function $\rho_2(\mathbf{r} - \mathbf{u}/2, \mathbf{r} + \mathbf{u}/2)$, which depends only on the relative distance u, can be calculated analytically [2]. However, this case cannot be achieved experimentally since for $\Omega = \omega$, the centrifugal force exactly balances the trapping force and the gas is not confined anymore. In a realistic model one must address the case $\Omega < \omega$, for which the trapping force overcomes the centrifugal one. The statistics of the roots then do not coincide with the standard results of the literature, and we must perform a study of this specific model.

To provide an intuitive understanding on how the roots of $f(\zeta)$ are distributed, we first show in Fig. 1 numerical results for a randomly generated field ψ. We take $k_B T = 500\hbar(\omega - \Omega)$, which is compatible with the condition $k_B T \ll 2\hbar\omega$ for the experimentally realistic value $\Omega = 0.999\omega$. On the first and second lines of Fig. 1, we show plots of $\ln(|\psi|^2)$ and $|\psi|^2$, respectively, for 3 values of μ. In all cases, the locations of the roots are clearly visible.

We now turn to an analytic study of the problem and we express the algebraic density of vortices $\rho_v(\mathbf{r})$ in terms of ψ and its derivatives [13]. A helpful simplification is to eliminate the Gaussian factor in Eq. (7), which does not change the vortex locations, and use $f(\zeta)$ rather than $\psi(\mathbf{r})$ as a random field. Using $\partial_y f(\zeta) = i\partial_x f(\zeta) = if'(\zeta)$, we obtain $\rho_v(\mathbf{r}) = |f'|^2\delta^{(2)}[f(\zeta)]$ making it obvious that all vortices have a positive charge in the LLL.

The expectation value of ρ_v gives the average vortex density $\bar{\rho}_v$, the correlation function $\langle\rho_v(\mathbf{r} - \mathbf{u}/2)\rho_v(\mathbf{r} + \mathbf{u}/2)\rangle$ gives the vortex pair distribution function ρ_2, etc. Using properties of Gaussian statistics, these quantities can be expressed in terms of expectation values of products of two fields, such as f, f^*, or their derivatives with respect to ζ [2]. The explicit result for $\bar{\rho}_v$ is

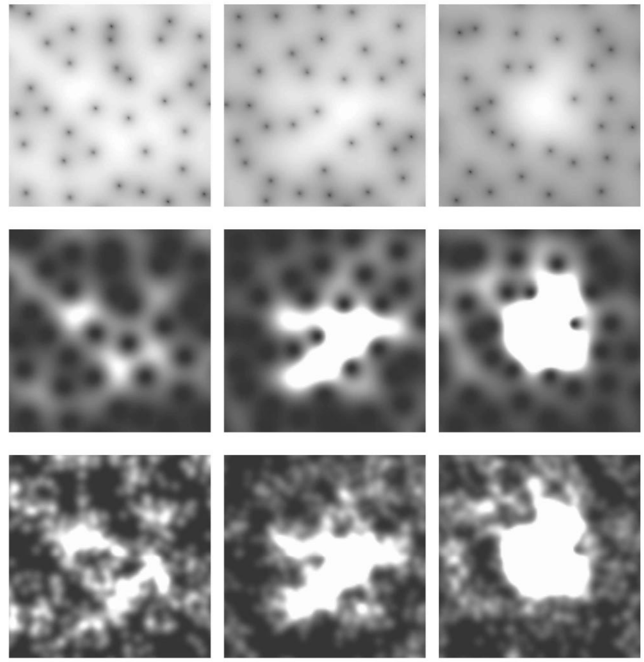

FIG. 1. Density plots for three randomly generated fields ψ, using the grand canonical ensemble for $k_B T = 500\epsilon_1$ [$\epsilon_1 = \hbar(\omega - \Omega)$]. Left column: $\mu = -10\epsilon_1$, mean number of particles $N = 1986$; middle column: $\mu = -\epsilon_1$, $N = 3396$; right column: $\mu = -0.1\epsilon_1$, $N = 8320$. First line: $\ln(|\psi(\mathbf{r})|^2)$. Second line: $|\psi(\mathbf{r})|^2$. Third line: simulation of a real experiment; N random atomic positions are generated according to the density $|\psi(\mathbf{r})|^2$, and a 129×129 pixel "camera" image is produced assuming that each atom produces a Gaussian spot of $\sigma = 0.2a_{\rm ho}$. The width of each image is $10a_{\rm ho}$.

$$\bar{\rho}_v(\mathbf{r}) = \frac{\langle f'^* f'\rangle}{\pi\langle f^* f\rangle} - \frac{\langle f^* f'\rangle\langle f f'^*\rangle}{\pi\langle f^* f\rangle^2}. \quad (8)$$

The expectation values $\langle\ldots\rangle$ are readily calculated using $\langle a_m^* a_{m'}\rangle = n_m\delta_{m,m'}$. Figure 2 gives the variation of $\bar{\rho}_v$ with the distance r from the trap center, for the temperature and the three values of the chemical potential used in Fig. 1.

The general expression for the pair distribution function ρ_2 is rather involved, and we give it only for small relative distances u. It vanishes quadratically with u, showing the effective repulsion between the vortices

$$\rho_2\left(\mathbf{r} - \frac{\mathbf{u}}{2}, \mathbf{r} + \frac{\mathbf{u}}{2}\right) = C(\mathbf{r})\frac{u^2}{8\pi^2} + O(u^3), \quad (9)$$

where C is a function of f, f^*, and their first and second derivatives [14].

We find that the experimentally relevant situation $\Omega < \omega$ leads to results similar to those of the standard random polynomial theory (formally corresponding to $\Omega = \omega$) in the vicinity of the trap center when the two following conditions are fulfilled. First, many eigenmodes have to be thermally populated:

$$k_B T \gg \hbar(\omega - \Omega). \tag{10}$$

Second, the low energy modes must have comparable populations, imposing a negative chemical potential $|\mu| \gg \hbar(\omega - \Omega)$. This is the situation depicted in the left column of Fig. 1 and in Fig. 2(a).

If $|\mu|$ is reduced for a fixed T, the mean number of atoms increases and an imbalance appears among the populations of the low energy modes. The vortex density is then depleted near the trap center, as can be seen in the middle and right columns of Fig. 1, and in Fig. 2(b) and 2(c). Eventually, for $|\mu| \ll \hbar(\omega - \Omega)$, the total population of the modes $m \neq 0$ saturates and a condensate forms in the mode $m = 0$. This condensate has no vortex and it expels the thermal vortices, which accumulate in a corona with a density locally exceeding $1/\pi$ [Fig. 2(c)].

The situation at the onset of condensation is relevant experimentally since it is tempting to increase the number of atoms in order to improve the experimental signal. To extend quantitatively our analysis to this regime, we must replace the grand canonical ensemble, which presents nonphysical, large fluctuations of the total particle number when a condensate is present, by the canonical ensemble. We then adapt our definition of the vortex distribution by projecting Eq. (4) on a subspace with a fixed total number of particles N, resulting in the following probability distribution of the field ψ:

$$P_c(\{\psi\}) \propto P(\{\psi\}) \frac{\|\psi\|^{2N}}{N!} e^{-\|\psi\|^2}, \tag{11}$$

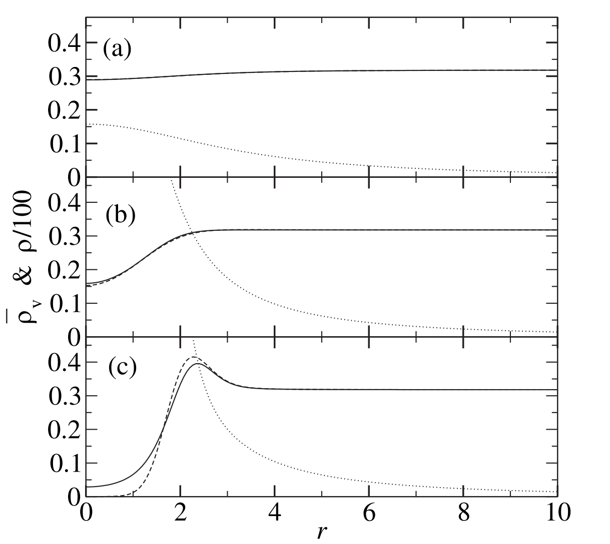

FIG. 2. Vortex ($\bar{\rho}_v$) and atomic (ρ) densities. Solid line: result for $\bar{\rho}_v$ in the grand canonical ensemble. Dashed line: result for $\bar{\rho}_v$ in the canonical ensemble with the same mean number of particles N. Dotted line: result for $\rho/100$ in the grand canonical ensemble. $k_B T = 500\epsilon_1$ with $\epsilon_1 = \hbar(\omega - \Omega)$. (a) $\mu = -10\epsilon_1$ ($N = 1986$). (b) $\mu = -\epsilon_1$ ($N = 3396$). (c) $\mu = -0.1\epsilon_1$ ($N = 8320$). The unit length is a_{ho}.

where $\|\psi\|^2 = \int |\psi|^2$. The key change with respect to the grand canonical ensemble is that the probability distribution of the field is no longer Gaussian, showing that the proposed physical system opens a new class of problems. We evaluate numerically the vortex density using a generating function technique [15]. In the trap center, one can also construct an exact mapping to a solvable problem, the calculation of the canonical partition function and occupation numbers for a 1D ideal Bose gas in a harmonic trap [16]. This leads to the exact expression

$$\bar{\rho}_v^{\,c}(\mathbf{r} = \mathbf{0}) = \frac{1 - e^{-\beta\epsilon_1}}{\pi}\left[\frac{1}{e^{\beta\epsilon_1} - 1} - n_1^c\right], \tag{12}$$

where n_1^c is the mean number of particles in the first excited mode $m = 1$ (with $\epsilon_1 = \hbar(\omega - \Omega)$) in the canonical ensemble [17]. As shown in Fig. 2(c), the correct canonical result drops to a much smaller value than the incorrect grand canonical result in the trap center. In the large N limit, one can show that $\bar{\rho}_v^c(\mathbf{0}) \sim N \exp[-(N + 1)\beta\epsilon_1]/\pi$, whereas the grand canonical prediction tends to zero only as $1/N$. Note that $\bar{\rho}_v^c(\mathbf{0})$ drops with N similarly to the probability $\exp(-N\beta\epsilon_1)$ of having an empty condensate mode, which is a natural condition to have a vortex in $\mathbf{r} = \mathbf{0}$.

How to observe the vortex density in practice? In current experiments with rotating interacting condensates, one measures the positions of the particles and the vortices appear as holes in the density profile [9]. This strategy can be used in regions of space where the mean density of particles ρ greatly exceeds the density of vortices $\bar{\rho}_v$, here $\sim 1/\pi$. To each vortex embedded in such a high density region will correspond a clearly identifiable hole in the particle distribution. The maximal density of the noncondensed fraction of the gas is $\sim k_B T/[\pi\hbar(\omega - \Omega)]$ close to the trap center, obtained for $|\mu| \ll \hbar(\omega - \Omega)$. Then $\rho \gg \bar{\rho}_v$ when Eq. (10) is satisfied. To indicate up to what distance from the trap center this condition holds, we have also plotted $\rho(r)/100$ in Fig. 2.

Conversely, do all the holes embedded in high density regions correspond to vortices? They may in principle correspond to a local minimum of $|\psi|^2$, where the field assumes a small but non zero value. Such "spurious" local minima can in general form in a noncondensed ideal Bose gas, which is subject to large density fluctuations due to the thermal bunching effect of the bosons. However, we can show that this phenomenon is absent in the LLL. We Taylor expand the field ψ to second order in $\mathbf{u} \equiv \mathbf{r} - \mathbf{r_0}$ around the location $\mathbf{r_0}$ of a stationary point of $|\psi|^2$, assuming that $\psi(\mathbf{r_0}) \neq 0$:

$$|\psi(\mathbf{r})|^2 = |\psi(\mathbf{r_0})|^2[1 + \mathbf{u} \cdot M\mathbf{u} + O(u^3)], \tag{13}$$

where the 2×2 matrix M is real symmetric. One then finds that the trace of M is

$$\mathrm{Tr}\, M = \mathrm{Re}\frac{\Delta\psi}{\psi} + \left|\frac{\mathbf{grad}\psi}{\psi}\right|^2 = -2, \tag{14}$$

where we used Eq. (7) and $\mathbf{grad}(|\psi|^2) = \mathbf{0}$ to get the last identity. Since its trace is <0, M cannot be positive and $|\psi|^2$ cannot have a local minimum at $\mathbf{r}_0$.

We exemplify this discussion by a Monte Carlo simulation of a real experiment for the parameters of Fig. 1. The results are shown on the last line of Fig. 1. Starting from the random fields ψ previously generated, we have produced these images by generating random positions of particles according to the distribution $|\psi|^2$, and by mimicking the finite imaging resolution of a real experiment. Local minima of the density are visible, and can be checked on the images of the first two lines to correspond to vortices. The visibility of the vortex pattern could be improved by increasing further the atom density. For a given ratio $k_B T/\hbar\omega$, this can be achieved by rotating the gas even faster, so that $(\omega - \Omega)/\omega$ decreases.

In real life, there are interactions between the particles, characterized by the 3D s-wave scattering length a. The interaction potential in the LLL is modeled by the pseudopotential $g\delta^{(2)}(\mathbf{r}_1 - \mathbf{r}_2)$, with $g = \sqrt{8\pi}\hbar\omega a/a_z$, where $a_z = \sqrt{\hbar/M\omega_z}$. We now derive a condition on a for the interactions to play a negligible role on the vortex distribution. Focusing on the quasihomogeneous regime $|\mu| \gg \hbar(\omega - \Omega)$, we set $\Omega = \omega$. The unperturbed occupation numbers then all have the same value n_0. The mean vortex density being uniform, we consider the first order correction in g to the vortex pair distribution function $\rho_2(\mathbf{u})$. Assuming $n_0 \gg 1$ we obtain

$$\frac{\delta\rho_2(\mathbf{u})}{\rho_2^{(0)}(\mathbf{u})} \simeq -\frac{\beta g}{2}\int d^2\mathbf{r}\left[\frac{\langle\rho_v(\mathbf{0})\rho_v(\mathbf{u})|\psi(\mathbf{r})|^4\rangle}{\rho_2^{(0)}(\mathbf{u})} - \langle|\psi(\mathbf{r})|^4\rangle\right], \tag{15}$$

where the average is taken over the unperturbed distribution. The right-hand side of Eq. (15) can be expressed analytically. It is maximal (in absolute value) in $u = 0$:

$$\lim_{u\to 0}\frac{\delta\rho_2(\mathbf{u})}{\rho_2^{(0)}(\mathbf{u})} = -\frac{3\beta g n_0^2}{16\pi} \propto \frac{g\rho}{|\mu|}, \tag{16}$$

whereas it drops as $u^4 e^{-u^2/2}$ at large distances. Interactions will then play a negligible role in the vortex distribution if the mean field energy $g\rho$ is much smaller than $|\mu|$. For the realistic numbers $k_B T = \hbar\omega/2$, $n_0 = 100$ and $a_z = 1\ \mu$m, this implies $|a| \lesssim 0.2$ nm.

In practice the scattering length can be tuned to such a low value using a Fano-Feshbach resonance. The experimental sequence should start with a larger value of a, to rapidly cool the gas in the LLL regime. Subsequently, a is set to its low value, and the residual elastic collisions ensure thermalization in the quasi-ideal regime. Finally the atom cloud is released from the trap, undergoes a free ballistic expansion to magnify its size, and the atom distribution is measured, revealing the vortex positions. Since the LLL modes are self-similar during the expansion [18], each result of this destructive measurement corresponds to a draw of a random polynomial.

To summarize, the rotating quasi-ideal Bose gas is a promising system to implement in practice the concepts developed in the theory of random polynomials. It also raises novel questions such as the influence of the non-Gaussian statistics for the polynomial coefficients when a condensate is present. Finally, we emphasize that our disordered vortex pattern appears in a nonsuperfluid system. An interesting future line of research is the transition to an ordered Abrikosov lattice when interactions are increased and the system becomes superfluid.

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[15] One defines the generating function $B(\theta) = \langle\exp[-\|\psi\|^2(1+e^{i\theta})]\rangle_P$. Then one computes the coefficient of B on the Fourier component $e^{iN\theta}$, which gives the normalization factor of $P_c(\{\psi\})$. A similar trick applies for the vortex density, so that $\bar{\rho}_{v,c} = \int_0^{2\pi} d\theta\,\bar{\rho}_{v,\theta}B(\theta)e^{-iN\theta}/\int_0^{2\pi} d\theta B(\theta)e^{-iN\theta}$ where $\bar{\rho}_{v,\theta}$ is taken from Eq. (8) replacing the mean occupation numbers n_m by $[\exp(\beta(\epsilon_m - \mu)) + \exp i\theta]^{-1}$.
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